EXERCISE – III HINTS & SOLUTIONS

Sol.1 **(i)** $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$

Order = 2
Degree = 3

(ii) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

Order = 2
Degree = 3

Sol.2 $\frac{\ell n(\sec x+\tan x)}{\cos x}\,dx = \frac{\ell n(\sec y+\tan y)}{\cos y}\,dy$

$\int \sec x\,\ell n(\sec x+\tan x)\,dx = \int \sec y\,\ell n(\sec y+\tan y)\,dy$

$[\ln(\sec x+\tan x)]^2 = [\ell n(\sec y+\tan y)]^2 + k$

Sol.3 $\frac{dy}{dx} + \frac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$

$\frac{y\,dy}{\sqrt{y^2-1}} + \frac{\sqrt{x^2-1}}{x}\,dx = 0$

$\frac{y\,dy}{\sqrt{y^2-1}} + \frac{x^2-1}{x\sqrt{x^2-1}}\,dx = 0$

$\frac{ydy}{\sqrt{y^2-1}} + \frac{x}{\sqrt{x^2-1}}\,dx - \frac{1}{x\sqrt{x^2-1}}\,dx = 0$

Integrate

$\sqrt{y^2-1} + \sqrt{x^2-1} - \sec^{-1}x = C$

Sol.4 $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

Put $x + y = t$

$1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - 1 = \sin t + \cos t$

$\frac{dt}{dx} = \sin t + \cos t + 1$

$\frac{dt}{\sin t+\cos t+1} = dx$

$\frac{\sec^2\frac{t}{2}dt}{1+\tan\frac{t}{2}} = 2dx$

$\ell n\left(1+\tan\frac{t}{2}\right) = x + C$

$\ell n\left(1+\tan\frac{x+y}{2}\right) = x + C$

Sol.5 **(a)** $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

$\frac{dy}{dx} = -2\cos\frac{x}{2}\sin\frac{y}{2}$

$\int \operatorname{cosec}\frac{y}{2}\,dy = -\int 2\cos\frac{x}{2}\,dx$

$\ell n\tan\frac{y}{4} = C - 2\sin\frac{x}{2}$

(b) $\sin x\frac{dy}{dx} = y\,\ell ny$

$\frac{dy}{y\,\ell ny} = \operatorname{cosec} x\,dx$

$\ell n(\ell xy) = \ln\left(\tan\frac{x}{2}\right) + C$

$\left(\frac{\pi}{2}, e\right) \Rightarrow \quad C = 0$

$\ell ny = \tan\frac{x}{2}$

$\Rightarrow y = e^{\tan\frac{x}{2}}$ Particular solution.

Sol.6 $e^{dy/dx} = x + 1$

$\frac{dy}{dx} = \ell n\,(x + 1)$

$\int dy = \int \ell n(x+1)\,dx$

$y = (x + 1) \int \ell n(x+1)\,dx$

$y = (x + 1)\,[\ln (x + 1)] + C$

$(0, 3) \Rightarrow C = 4$

$y = (x + 1)\,\ell n\,(x + 1) - x - 1 + 4$

$y = (x + 1)\,\ell n\,(x + 1) - x + 3$ Particular soluton

Sol.7 (a) $\frac{-dP}{dt} = k\,(P - 1000)$

$\ln (P - 1000) = -kt + c$

$P = 1000 + c_1\,e^{-kt}$

$t = 0 \Rightarrow P = 2500$

$\Rightarrow c_1 = 1500$

$P = 1000 + 1500\,e^{-kt}$

(b) $t = 10 \Rightarrow P = 1900$

$900 = 1500\,e^{-k(10)}$

$\Rightarrow k = \frac{1}{10}\,\ln\left(\frac{5}{3}\right)$

$\Rightarrow P = 1000 + 1500\,e^{-kt}$

$1500 = 1000 + 1500\,e^{-kt}$

$\Rightarrow e^{kt} = 3 \Rightarrow kt = \ln 3$

$t = \frac{\ln 3}{k} = 10\,\frac{\ln 3}{\ln\left(\frac{5}{3}\right)}$

$= 10.\,\ln_{5/3} 3$

Sol.8 $\frac{-dm}{dt} = Km$

$\ell n\, m = -kt + \ell nc$

$\frac{m}{c} = e^{-kt} \qquad \Rightarrow m = ce^{-t}$

at $t = 0$, $m = m_0 \qquad \Rightarrow C = m_0$

$m = m_0 e^{-kt}$

at $t = t_0$, $m = m_0\left(1-\frac{\alpha}{100}\right)$

$m_0\left(1-\frac{\alpha}{100}\right) = m_0\,e^{-kt_0} \Rightarrow k = \frac{-1}{t_0}\,\ell n\left(1-\frac{\alpha}{100}\right)$

$m = m_0\,e^{-kt}$

Where

$K = \frac{-1}{t_0}\,\ell n\left(1-\frac{\alpha}{100}\right)$

Sol.9 L_N (Length of the normal) $= y\sqrt{1+m^2}$

$y\sqrt{1+m^2} = k \Rightarrow 1 + m^2 = \frac{k^2}{y^2}$

$m^2 = \frac{k^2-y^2}{y^2} \Rightarrow m = \pm\frac{\sqrt{k^2-y^2}}{y}$

$y\,\frac{dy}{dx} = \pm\sqrt{k^2-y^2} \Rightarrow \frac{y\,dx}{\sqrt{k^2-y^2}} = \pm\,dx$

$-\sqrt{k^2-y^2} = \pm x + C$

$(0, k) \qquad \Rightarrow C = 0$

$-\sqrt{k^2-y^2} = \pm x$

$k^2 - y^2 = x^2$

$x^2 + y^2 = k^2$ Circle

Sol.10 $\frac{y\sqrt{1+m^2}}{m} + \frac{y}{m} = kxy$

$\frac{\sqrt{1+m^2}+1}{m} = kx$

$\sqrt{1+m^2} = (kxm - 1)$

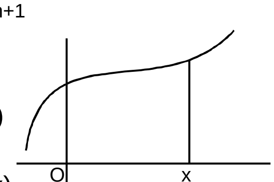
$1 + m^2 = k^2x^2m^2 + 1 - 2kxm$

$m = 0\;;\; m = \frac{-2kx}{1-k^2x^2} \Rightarrow \frac{dy}{dx} = \frac{-2kx}{1-k^2x^2}$

$dy = \frac{1}{k}\left[\frac{-2k^2x}{1-k^2x^2}\right]dx$

$y = \frac{1}{k}\,\ell n\,|(1 - k^2x^2)| + \ell nC$

Sol.11 $A = \int_0^x f(x)\,dx = k\,[f(x)]^{n+1}$



Use Leibitz

$f(x)=k\,(n + 1)\,[f(x)]^n.f'(x)$

$\frac{1}{k(n+1)} = [f(x)]^{n-1}\,.\,f'(x)$

$\frac{x}{k(n+1)} = \frac{[f(x)]^n}{n} + C$

$x = 0,\ f(0) = 0 \quad \Rightarrow \quad C = 0$

$x = 1,\ f(1) = 1 \quad \Rightarrow \quad k = \frac{1}{n+1}$

$[f(x)]^n = x$

$f(x) = x^{1/n}$

Sol.12 Given that

$$y\,\frac{\sqrt{1+m^2}}{m} = \sqrt{x^2+y^2}$$

$y^2 (1 + m^2) = m^2 (x^2 + y^2)$

$y^2 = m^2x^2$

$$m = \pm\,\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} \qquad \frac{dy}{dx} = \frac{-y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x} \qquad \frac{dy}{y} = -\frac{dx}{x}$$

$\ell ny = \ell nx + ek \quad \ell ny = -\ell nx + k$

$$\frac{y}{x} = k' \qquad xy = k'$$

$y = k'x$

Sol.13 (a) $$\frac{dy}{dx} = \frac{x^2+xy}{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{1+y/x}{1+(y/x)^2}$$

Put $y = tx \quad \Rightarrow \frac{dy}{dx} = t + x\,\frac{dt}{dx}$

$$t + x\,\frac{dt}{dx} = \frac{1+t}{1+t^2} \Rightarrow x\,\frac{dt}{dx} = \frac{1+t}{1+t^2} - t$$

$$x\,\frac{dt}{dx} = \frac{1+t-t-t^3}{1+t^2} \Rightarrow \frac{(1+t^2)dt}{(1-t^3)} = \frac{dx}{x}$$

$$\frac{t^2}{1-t^3}\,dt + \frac{1}{1-t^3}\,dt = \frac{dx}{x}$$

$$\frac{t^2}{1-t^3}\,dt + \left(\frac{1-t^2+t^2}{1-t^3}\right)dt = \frac{dx}{x}$$

$$\left(\frac{2t^2}{1-t^3}\right)dt + \frac{(1-t)(1+t)dt}{(1-t)(1-t^2+t)} = \frac{dx}{x}$$

$$\frac{2}{3}\left[\frac{3t^2}{1-t^3}\right]dt + \frac{1}{2}\left[\frac{2t+1}{t^2+t+1}\right]dt + \frac{1}{2}\,\frac{dt}{t^2+t+1} = \frac{dx}{x}$$

Integrate both the side

$$-\frac{2}{3}\,\ell n\,(1-t^3) + \frac{1}{2}\,\ell n\,(t^2+t+1)$$

$$+\frac{1}{2.\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) = \ell n\,x + C$$

After putting value of 't'

$C\,(x-y)^{2/3}\,(x^2+xy+y^2)^{1/6}$

$$= \exp\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2y}{x\sqrt{3}}\right]$$

(b) $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

$$\frac{dy}{dx} = \frac{x^3-3xy^2}{y^3-3x^2y}$$

$$\frac{dy}{dx} = \frac{1-3\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)^3 - 3\left(\frac{y}{x}\right)}$$

Put $y = tx \quad \Rightarrow \frac{dy}{dx} = t + x\,\frac{dt}{dx}$

$$t + x\,\frac{dt}{dx} = \frac{1-3t^2}{t^3-3t}$$

$$x\,\frac{dt}{dx} = \frac{1-3t^2-t^4+3t^2}{t(t^2-3)}$$

$$\frac{t\,(t^2-3)}{1-t^4}\,dt = \frac{dx}{x}$$

After integration

$y^2 - x^2 = c\,(y^2 + x^2)^2$

Sol.14 $x = |y\,\sqrt{1+m^2}\,|$

$y\,\sqrt{1+m^2} = \pm\,x$

$$\sqrt{1+m^2} = \pm\,\frac{x}{y} \Rightarrow 1 + m^2 = \frac{x^2}{y^2}$$

$$m^2 = \frac{x^2-y^2}{y^2} \Rightarrow \frac{dy}{dx} = \pm\sqrt{\frac{x^2-y^2}{y^2}}$$

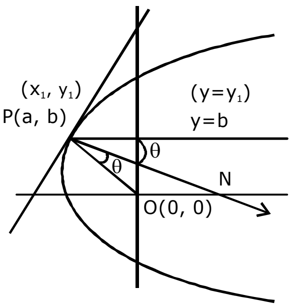
Put $\quad y = t\,x \Rightarrow \frac{dy}{dx} = t + x\,\frac{dt}{dx}$

$$\frac{t\,dt}{\sqrt{1-t^2}-t^2} = \frac{dx}{x}$$

After integration

$$\frac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \ell n\left[(y \pm \sqrt{y^2-x^2})\frac{c^2}{x^3}\right]$$

Sol.15 Equation of PN (angle bisector of emanating and reflected ray) $\dfrac{y-b}{1} = \pm \dfrac{bx-ay}{\sqrt{a^2+b^2}}$



$$\frac{b}{\sqrt{a^2+b^2}}\,x - \left(\frac{a+\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}}\right) y + b = 0$$

$$\frac{y}{x+\sqrt{a^2+b^2}} = \frac{dy}{dx}$$

$$\frac{ydx - xdy}{y^2} = \frac{\sqrt{x^2+y^2}}{y^2}\,dx$$

$$\int \frac{d(x/y)}{\sqrt{\left(\frac{x}{y}\right)^2+1}} = \int \frac{dy}{y}$$

$$\ln\left(\frac{x}{y}+\sqrt{\frac{x^2}{y^2}+1}\right) = \ln y + c$$

$$x + \sqrt{x^2+y^2} = ky^2$$

$1 = k^2y^2 - 2kx$ (Parabola)

Sol.16 $Y - y = m(X - x)$

$Y - y - m(X - x) = 0$

$$\left|\frac{-y+mx}{\sqrt{1+m^2}}\right| = x$$

$(mx - y)^2 = x^2(1 + m^2)$

$m^2x^2 - 2mxy + y^2 = x^2 + x^2m^2$

$2mxy = y^2 - x^2$

$$m = \frac{y^2-x^2}{2xy}$$

$$\frac{dy}{dx} = \frac{\frac{y^2}{x^2}-1}{\frac{2y}{x}} \qquad \text{Put } y = tx$$

$$t + \frac{xdt}{dx} = \frac{t^2-1}{2t}$$

$$x\frac{dt}{dx} = \frac{t^2-1-2t^2}{2t}$$

$$\frac{2t}{-t^2-1}\,dt = \frac{dx}{x}$$

$\ell n\,(t^2 + 1) = -\,\ell nx + \ell nC$

$x(t^2 + 1) = C$

$$\left(\frac{y^2+x^2}{x}\right) = C$$

$C = 2$ (1, 1) Passes

$y^2 + x^2 = 2x$

$x^2 + y^2 - 2x = 0$

Sol.17 $|ym| = \dfrac{x+y}{2}$

+ve $\quad y\dfrac{dy}{dx} = \dfrac{x+y}{2}$

$$2\frac{dy}{dx} = \frac{x+y}{y}$$

Put $y = tx$

$$2\left[t + x\frac{dt}{dx}\right] = \frac{1+t}{t}$$

$$2t + 2x\frac{dt}{dx} = \frac{1+t}{t} \Rightarrow 2x\frac{dt}{dx} = \frac{1+t-2t^2}{t}$$

$$\frac{2t}{1+t-2t^2}\,dt = dx$$

After integrating and then pases thorugh (1, 0) will get

$(y - x)^2(x + 2y) = 1$

Sol.18 $y^2 = a - x \qquad y^3\dfrac{dy}{dx} + x + y^2 = 0$

$$2y\frac{dy}{dx} = \frac{da}{dx} - 1$$

$$y^3\frac{dy}{dx} + x + y^2 = 0$$

$$y^2\left(2y\frac{dy}{dx}\right) + 2x + 2y^2 = 0$$

$$(a - x)\left(\frac{da}{dx} - 1\right) + 2x + 2(a - x) = 0$$

$$(a - x)\left(\frac{da}{dx} - 1\right) + 2a = 0$$

$$\frac{da}{dx} - 1 = \frac{-2a}{a - x} \Rightarrow \frac{da}{dx} = \frac{-a - x}{a - x}$$

Put $a = tx$

$$\frac{da}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = \frac{-1 - a/x}{a/x - 1}$$

$$x\frac{dt}{dx} = \frac{-1 - t}{t - 1} - t$$

$$x\frac{dt}{dx} = \frac{-1 - t - t^2 + t}{t - 1}$$

$$\frac{t - 1}{-1 - t^2}\,dt = \frac{dx}{x} \Rightarrow \frac{1 - t}{1 + t^2}\,dt = \frac{dx}{x}$$

$$\left(\frac{1}{1 + t^2}\right) - \frac{1}{2}\left(\frac{2t}{1 + t^2}\right)dt = \frac{dx}{x}$$

$$\tan^{-1}t - \frac{1}{2}\,\ell n\,(1 + t^2) = \ell nx + \ell nC$$

$$\tan^{-1}\frac{a}{x} - \frac{1}{2}\,\ell n\left(\frac{a^2 + x^2}{x^2}\right) = \ell nx + \ell nC$$

$$\tan^{-1}\frac{a}{x} - \frac{1}{2}\,\ell n\,(a^2 + x^2) = \ell nC$$

$$\frac{1}{2}\,\ell n\,(a^2 + x^2) - \tan^{-1}\frac{a}{x} = k$$

Where $a = y^2 + x$

Sol.19 $\left[x\cos\frac{y}{x} + y\sin\frac{y}{x}\right] y = \left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right] x\,\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\left[x\cos\frac{y}{x} + y\sin\frac{y}{x}\right]y}{\left[y\sin\frac{y}{x} - x\cos\frac{y}{x}\right]x}$$

$$\frac{dy}{dx} = \frac{\left[\cos\frac{y}{x} + \left(\frac{y}{x}\right)\sin\frac{y}{x}\right]\frac{y}{x}}{\left[\frac{y}{x}\sin\frac{y}{x} - \cos\frac{y}{x}\right]}$$

$y = tx$

$$x\frac{dt}{dx} = \frac{t[\cos t + t\sin t]}{[t\sin t - \cos t]} - t$$

$$x\frac{dt}{dx} = \frac{t\cos t + t^2\sin t - t^2\sin t + t\cos t}{t\sin t - \cos t}$$

$$\frac{t\sin t - \cos t}{2t\cos t}\,dt = \frac{dx}{x}$$

$-\ell n\,(t\cos t) = 2\ell nx + \ell nC$

$\ell n\,(t\cos t)\,x^2 = \ell nk$

$(t\cos t)\,x^2 = k'$

$$\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)x^2 = k' \quad \Rightarrow \quad xy\cos\left(\frac{y}{x}\right) = k'$$

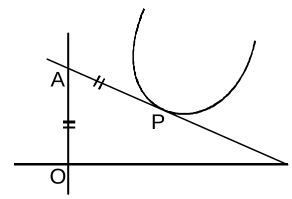
Sol.20 $Y - y = m\,(X - x)$

$A\,(0, y - mx)$

$O\,(0, 0)$

$P\,(x, y)$

$OA = PA$



$|y - mx| = \sqrt{x^2 + m^2x^2}$

$y^2 + m^2x^2 - 2mxy = x^2 + m^2x^2$

$$\frac{y^2 - x^2}{2xy} = m$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = tx$

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + \frac{xdt}{dx} = \frac{t^2 - 1}{2t}$$

$$x\frac{dt}{dx} = \frac{t^2 - 1}{2t}$$

$$\frac{2t}{1 + t^2}\,dt = \frac{-dx}{x}$$

$\ell n\,(1 + t^2) + \ell nx = \ell nC$

$(1 + t^2)\,x = C \qquad x^2 + y^2 = Cx$

Sol.21 $\dfrac{dy}{dx} = \dfrac{(x + y + 1)}{x - y}$

Let $x = X + h \Rightarrow dx = dX$

$y = Y + k \Rightarrow dy = dY$

$$\frac{dY}{dX} = \frac{(X + h) + (Y + k) + 1}{(X + h) - (Y + k)}$$

$$\frac{dY}{dX} = \frac{(X+Y)+(h+k+1)}{(X-Y)+(h-k)}$$

h + k + 1 = 0 & h – k = 0

$$\Rightarrow h = -\frac{1}{2},\ k = -\frac{1}{2}$$

$$\frac{dY}{dX} = \frac{X+Y}{X-Y} \qquad \text{Put } Y = tX$$

$$X\frac{dt}{dX} = \frac{1+t}{1-t} - t \ \Rightarrow\ \frac{1-t}{1+t^2}\,dt = \frac{dX}{X}$$

$$\tan^{-1} t - \frac{1}{2}\,\ell n\,(1+t^2) = \ell nx + \ell nC$$

$$\tan^{-1} t - \frac{1}{2}\,\ell n\,(y^2+x^2) = \ell nC$$

$$\tan^{-1}\frac{y}{x} = \ell nC\sqrt{y^2+x^2}$$

$$\tan^{-1}\left(\frac{y+\frac{1}{2}}{x+\frac{1}{2}}\right) = \ell n\,C\sqrt{\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2}$$

$$\tan^{-1}\left(\frac{2y+1}{2x+1}\right) = \ell nC\sqrt{x^2+y^2+x+y+\frac{1}{2}}$$

Sol.22 $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$

$$2\frac{dy}{dx} = \frac{x+y+1}{x+y+1+\frac{1}{2}} \qquad \text{Put } x+y+1 = t$$

$$2\left[\frac{dt}{dx}-1\right] = \frac{t}{t+\frac{1}{2}} \quad ; \quad 1+\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - 1 = \frac{t}{2t+1}$$

$$\frac{dt}{dx} = \frac{3t+1}{2t+1}$$

$$\left(\frac{2t+1}{3t+1}\right)dt = dx$$

$$\frac{2}{3}\left[\frac{3t+\frac{3}{2}}{3t+1}\right] = dt = dx$$

$$\frac{2}{3}\left[\frac{3t+1+\frac{1}{2}}{3t+1}\right]dt = dx$$

$$\frac{2}{3}\left[1+\frac{1}{2(3t+1)}\right]dt = dx$$

$$\frac{2}{3}t + \frac{1}{9}\,\ell n\,(3t+1) = x + C$$

$(3t + 1) = ke^{3(x-2y)}$

$3(x + y + 1) + 1 = ke^{3(x-2y)}$

$$x + y + \frac{4}{3} = k'e^{3(x-2y)}$$

Sol.23 tangent $Y - y = m\,(X - x)$

Normal $Y - y = \dfrac{-1}{m}\,(X - x)$

$\perp^r$ from origin to tangent $= \perp^r$ to Normal

$$\left|\frac{-y+mx}{\sqrt{1+m^2}}\right| = \left|\frac{-y-x/m}{\sqrt{1+1/m^2}}\right|$$

$(y - mx)^2 = (my + x)^2$

$\Rightarrow\ m^2(x^2 - y^2) - 4m\,xy + y^2 - x^2 = 0$

$$m = \frac{dy}{dx} = \frac{2xy \pm (x^2+y^2)}{x^2-y^2}$$

$$+ve \quad \frac{dy}{dx} = \frac{(x+y)^2}{x^2-y^2} = \frac{x+y}{x-y}\ ;$$

$$-ve \quad \frac{dy}{dx} = -\frac{(x-y)}{x+y}$$

Homogenous $y = tx$

$$x\frac{dt}{dx} = \frac{1+t^2}{1-t}$$

$$\frac{1}{1+t^2} - \frac{1}{2}\left(\frac{2t}{1+t^2}\right)dt = \frac{dx}{x}$$

$$\tan^{-1}\frac{1}{1+t^2} - \frac{1}{2}\ln(1+t^2) = \ln x + \ln k$$

$$\sqrt{x^2+y^2} = c\ e^{\tan^{-1} y/x}$$

Similarly if we take –ve sign

$$\sqrt{x^2+y^2} = c\ e^{-\tan^{-1} y/x}$$

Sol.24 $y = y(t)$

$$\frac{dy}{dt} + 2yt = t^2$$

$$\text{I.F.} = e^{\int 2t\,dt} = e^{t^2}$$

$$y e^{t^2} = \frac{\int t^2 e^{t^2} dt}{e^{t^2}}$$

$$\lim_{t\to\infty} \frac{y}{t} = \lim_{t\to\infty} \frac{\int t^2 e^{t^2} dt}{t\, e^{t^2}} = \lim_{t\to\infty} \frac{t^2 e^{t^2}}{e^{t^2} + 2t^2 e^{t^2}}$$

$$= \lim_{t\to\infty} \frac{t^2}{2t^2+1} = \frac{1}{2}$$

Sol.25 $(1 - x^2)\dfrac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

$$\text{I.F.} = e^{\int \frac{2x}{1-x^2}dx} = e^{-\ell n(1-x^2)} = \frac{1}{1-x^2}$$

$$\frac{y}{1-x^2} = \int \frac{x}{(1-x^2)^{3/2}}\,dx$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C$$

$$y = C(1 - x^2) + \sqrt{1-x^2}$$

Sol.26 Tangent

$$Y - y = \frac{dy}{dx} \quad (x - x)$$

$$\text{put } X = 0 \Rightarrow Y = \left(y - x\frac{dy}{dx}\right)$$

$$A = \frac{1}{2}x^2$$

$$\left|\frac{1}{2}\left[2y - x\frac{dy}{dx}\right].x\right| = \frac{1}{2}x^2$$

$$2y - x\frac{dy}{dx} = \pm x$$

$$\frac{dy}{dx} \quad \frac{2}{x} y = \pm 1$$

$$\text{IF} = e^{\int -\frac{2}{x}dx} = \frac{1}{x^2}$$

$$\left(\frac{y}{x^2}\right) = \int \pm \frac{1}{x^2}\,dx$$

$$\frac{y}{x^2} = \pm \frac{1}{x} + c$$

$$y = cx^2 \pm x$$

Sol.27 $Y - y = m(X - x)$

$X = 0,\ Y = y - mx$

$y - mx = x^2$

$$y - x\frac{dy}{dx} = x^2$$

$$\frac{ydx - xdy}{x^2} = dx$$

$$d\left(\frac{y}{x}\right) = -dx$$

$$\frac{y}{x} = -x + C$$

$$y = cx - x^2$$

Sol.28 $x\dfrac{dy}{dx} - y = 2x^2 \text{ cosec } 2x$

$$\frac{dy}{dx} - \frac{y}{x} = 2x \text{ cosec } 2x$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = e^{-\ell nx} = \frac{1}{x}$$

$$\frac{y}{x} = \int 2\text{cosec}x\,dx$$

$$\frac{y}{x} = \ln \tan x + C$$

$$y = cx + x\,\ell n \tan x$$

Sol.29 $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

Put $\tan^{-1}y = t \Rightarrow \left(\dfrac{1}{1+y^2}\right) dy = dt$

$dx = (t - x)\,dt \Rightarrow \dfrac{dx}{dt} + x = t$

I.F. = $e^{\int 1.dt} = e^t$

$x e^t = \int t e^t \, dt$

$xe^t = te^t - e^t + c$

$x = t - 1 + ce^{-t}$

$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$

Sol.30 y – Intercept of tangent = y – mx

$x(y - mx) = \pm a^2$

$xy - x^2 \frac{dy}{dx} = \pm a^2$

$x^2 \frac{dy}{dx} = xy \mp a^2$

$\frac{dy}{dx} = \frac{y}{x} \mp \frac{a^2}{x^2}$

$\frac{dy}{dx} - \frac{y}{x} = \mp \frac{a^2}{x^2}$

I.F. = $e^{-\int \frac{1}{x} dx} = e^{-\ell nx} = \frac{1}{x}$

$\frac{y}{x} = \pm \frac{a^2}{x^3} \, dx$

$\frac{y}{x} = \pm \frac{a^2}{2x^2} + C$

$y = Cx \pm \frac{a^2}{2x}$

Sol.31 $\frac{d}{dx}(x\, f(x)) \le -k\, f(x)$

$x f'(x) + f(x) \le -k\, f(x)$

$xf'(x) + (1 + k) f(x) \le 0$

multiply by x^k

$x^{k+1} f'(x) + (k + 1) x^k f(x) \le 0$

$\frac{d}{dx}(x^{k+1}. f(x)) \le 0$

Let $F(x) = x^{k+1} f(x)$

F(x) is decreasing for $x \ge 2$

$F(x) \le F(2)$

$F(x) \le A$

$x^{k+1} f(x) \le A$

$f(x) \le A. x^{-1-k}$

Sol.32 $f(x) = -\int_0^x f(t) \tan t \, dt + \int_0^x \tan(t - x) \, dt$

put $t - x = z \Rightarrow dt = dz$

$f(x) = -\int_0^x f(t) \tan t \, dt + \int_{-x}^0 \tan z \, dz$

Now use leibnitz

$f'(x) = -f(x) \tan x - \tan x$

$\frac{dy}{dx} + y \tan x = -\tan x$

I.F. = $e^{\int \tan x \, dx} = \sec x$

$y(\sec x) = -\int \tan x \sec x \, dx = -\sec x + c$

$y = c \cos x - 1$

$y(0) = 0 \Rightarrow c = 1$

$y = \cos x - 1$

Sol.33 Given that $F(x) = \int f(x) dx \Rightarrow f(x) = F'(x)$

$F'(x) + \cos x.\, F(x) = \frac{\sin 2x}{(1 + \sin x)^2}$

$\frac{dy}{dx} + y \cos x = \frac{\sin 2x}{(1 + \sin x)^2}$

I.F. = $e^{\int \cos x \, dx} = e^{\sin x}$

$y(e^{\sin x}) = \int \frac{2 \sin \cos x}{(1 + \sin)^2} e^{\sin x} \, dx$

put sinx = t

$= 2 \int \frac{t}{(1 + t)^2} e^t \, dt$

$y(e^{\sin x}) = \frac{2e^{\sin x}}{1 + \sin x} + c$

$F(x) = \frac{2}{1 + \sin x} + c\, e^{-\sin x}$

$F'(x) = \frac{-2 \cos x}{(1 + \sin x)^2} - c\, e^{-\sin x} \cos x$

Sol.34 Rate at which fertilizer added = 1 × 1=1 gm/min.
volume of solution at time t = 100 + (1 – 3) t
= 100 – 2t

$$\Rightarrow \frac{dy}{dt} = 1 - \left(\frac{3y}{100-2t}\right)$$

$$\frac{dy}{dt} + \left(\frac{3}{100-2t}\right)y = 1$$

$$\Rightarrow y = (100-2t) + c\,(100-2t)^{3/2}$$

$$\text{at } t = 0,\ y = 0 \Rightarrow c = -\frac{1}{10}$$

$$y = (100-2t) - \frac{1}{10}(100-2t)^{3/2}$$

$$\frac{dy}{dt} = 0 \Rightarrow t = 27\frac{7}{9}$$

Sol.35 Rate at which salt added = 4 × 1 = 4 gm/min.
volume of solution at time t = 40 + (4–2)t = 40+2t

$$\frac{dy}{dt} = 4 - \left(\frac{y}{40+2t}\right)2$$

$$\frac{dy}{dt} = 4 - \frac{y}{200+t}$$

Sol.36 $(x - y^2)\,dx + 2xy\,dy = 0$

$$\frac{dy}{dx} = \frac{y}{2x} - \frac{1}{2y}; \text{ put } y^2 = t$$

$$2y\frac{dy}{dx} = \frac{y^2}{x} - 1$$

$$2y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} = \frac{t}{x} - 1$$

$$\frac{dt}{dx} - \frac{t}{x} = -1$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = -\frac{1}{x}$$

$$\frac{t}{x} = -\int \frac{1}{x}dx$$

$$\frac{t}{x} = -\ell nx - \ell nC$$

$$\frac{y^2}{x} + \ell n\,(xC) = 0$$

$$y^2 + x\,\ell n\,(xc) = 0$$

Sol.37 $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\,e^x \sec y$

$$\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\,e^x$$

$$\sin y = t \qquad \Rightarrow \cos y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} - \frac{t}{1+x} = (1+x)\,e^x$$

$$\text{I.F.} = e^{-\int\frac{dx}{1+x}} = \frac{1}{1+x}$$

$$\frac{t}{1+x} = \int e^x\,dx$$

$$\frac{t}{1+x} = e^x + C$$

$$\frac{\sin y}{1+x} = e^x + C$$

$$\sin y = (1+x)(e^x + C)$$

Sol.38 $\frac{dy}{dx} = \frac{e^y}{x^2} - \frac{1}{x}$

$$\frac{1}{e^y}\frac{dy}{dx} = \frac{1}{x^2} - \frac{e^{-y}}{x}$$

$$e^{-y} = t$$

$$e^{-y}\frac{dy}{dx} = -\frac{dt}{dx}$$

$$-\frac{dt}{dx} = \frac{1}{x^2} - \frac{t}{x}$$

$$\frac{dt}{dx} - \frac{t}{x} = \frac{-1}{x^2}$$

$$\text{I.F.} = \frac{1}{x} \Rightarrow \frac{t}{x} = -\int \frac{1}{x^3}dx$$

$$\frac{t}{x} = \frac{1}{2x^2} + C$$

$$2xe^{-y} = 1 + Cx^2$$

Sol.39 $\left(\frac{dy}{dx}\right)^2 - (x + y)\frac{dy}{dx} + xy = 0$

$$\frac{dy}{dx} = \frac{(x+y) \pm \sqrt{(x+y)^2 - 4xy}}{2}$$

$$\frac{dy}{dx} = \frac{(x+y) \pm (x-y)}{2}$$

+ve $\quad \frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + C$

–ve $\quad \frac{dy}{dx} = y \Rightarrow \ell ny = x + C$

$y = ke^x$

Sol.40 $\frac{dy}{dx} = \frac{y^2 - x}{2y(x+1)}$

$$2y\frac{dy}{dx} = \frac{y^2}{(x+1)} - \frac{x}{(x+1)}$$

Put $y^2 = t \quad \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

from $\frac{dt}{dx} - \frac{t}{(x+1)} = \frac{-x}{(x+1)}$

I.F. $= \frac{1}{(x+1)}$

$$\frac{t}{(x+1)} = -\int \frac{x}{(x+1)^2}dx$$

$$= -\int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2}$$

$$\frac{y^2}{(x+1)} = -\ell n\,(x + 1) - \frac{1}{(x+1)} + \ell nC$$

$$y^2 = (x + 1)\,\ell n\frac{C}{(x+1)} - 1$$

Sol.41 $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

$$\frac{dy}{dx} = \frac{e^{2x}}{e^y} - e^x$$

$$e^y\frac{dy}{dx} + e^x e^y = e^{2x}$$

$e^y = t \quad \Rightarrow e^y\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + e^x t = e^{2x}$$

I.F. $= e^{\int e^x dx} = e^{e^x}$

$$t\,e^{e^x} = \int e^{e^x}.\ \underset{\substack{e^x = z \\ e^x dx = dz}}{e^{2x}dx}$$

$$= \int ze^z\,dz$$

$$t\,e^{e^x} = ze^z - e^z + C$$

$$t\,e^{e^x} = e^x e^{e^x} - e^{e^x} + C$$

$e^y = (e^x - 1) + C\ \text{exp.}\,(-e^x)$

Sol.42 $y\frac{dy}{dx}\sin x = \cos x \sin x - y^2 \cos x$

$y^2 = t \quad \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{dt}{dx}$

$$\frac{1}{2}\frac{dt}{dx}\sin x = \cos x \sin x - t\cos x$$

$$\frac{dt}{dx} + 2t\cot x = 2\cos x$$

I.F. $= e^{\int 2\cot x\,dx} = (\sin x)^2$

$$t(\sin x)^2 = 2\int \cos x(\sin x)^2 dx$$

$$t(\sin x)^2 = \frac{2(\sin x)^3}{3} + C$$

$$y^2 = \frac{2}{3}(\sin x) + \frac{C}{(\sin x)^2}$$